

U-Duality, D-Branes and Black Hole Emission Rates: Agreements and Disagreements

Fay Dowker^a, David Kastor^{b,1} and Jennie Traschen^{b,2}

^a *Blackett Laboratory,
Imperial College of Science, Technology and Medicine,
London, SW7 2BZ.
Internet: dowker@ic.ac.uk*

^b *Department of Physics and Astronomy
University of Massachusetts
Amherst, MA 01003-4525*

¹ *Internet: kastor@phast.umass.edu*

² *Internet: lboo@phast.umass.edu*

Abstract

An expression for the spacetime absorption coefficient of a scalar field in a five dimensional, near extremal black hole background is derived, which has the same form as that presented by Maldacena and Strominger, but is valid over a larger, U-duality invariant region of parameter space and in general disagrees with the corresponding D-brane result. We develop an argument, based on D-brane thermodynamics, which specifies the range of parameters over which agreement should be expected. For neutral emission, the spacetime and D-brane results agree over this range. However, for charged emission, we find disagreement in the ‘Fat Black Hole’ regime, in which charge is quantized in smaller units on the brane, than in the bulk of spacetime. We indicate a possible problem with the D-brane model in this regime. We also use the Born approximation to study the high frequency limit of the absorption coefficient and find that it approaches unity, for large black hole backgrounds, at frequencies still below the string scale, again in disagreement with D-brane results.

February, 1997

1. Introduction

Progress made in the past year has led towards a possible understanding of the nature of black hole microstates within string theory (see *e.g.* the reviews [1,2] and references therein). Counting arguments based on the weak coupling D-brane description of string solitons have been shown to reproduce the Bekenstein-Hawking formula for the entropy of certain extremal and near-extremal black holes. The D-brane description also leads to a manifestly unitary prescription for calculating the rate of absorption/emission from the D-brane system [3], reproducing the characteristic black body form of the Hawking emission spectrum. A number of authors [4,5,6,7,8,9,10,11,12,13,14,15,16] have gone on to compare in more detail the spectra of emitted radiation calculated for the D-brane system with that calculated using semiclassical spacetime methods.

In the present paper we continue this work of detailed comparison between the D-brane and spacetime calculations. We present a number of new results, including a U-duality invariant (covariant) expression for the spacetime emission rate of neutral (charged) particles, together with a discussion based on D-brane thermodynamics of the regime in which we would expect to obtain agreement between this spacetime expression and D-brane results; a discussion of the expected higher energy behavior of the spacetime emission rate; and, most interestingly, a regime of disagreement between the D-brane and spacetime calculations for emission of charged particles. This regime of disagreement occurs in the ‘fat black hole’ region of parameter space, in which charge is quantized in smaller units on the brane, than in the bulk of spacetime [17,18]. It is left as an open question whether this disagreement can be resolved through a clearer understanding of emission processes in the ‘fat black hole’ limit.

One might be tempted to dismiss these discrepancies as irrelevant since it was argued in [7] that the spacetime calculation of emission from a black hole is only valid for black holes larger than the string scale whereas the perturbative D-brane calculation is only valid for black holes smaller than the string scale. This would mean that there should be no reason to expect agreement between the two. Nevertheless, agreement, where it has been found, has been taken as further evidence for the general idea that D-branes provide the correct microscopic description of black holes in string theory. It is important, then, not only to point out when unexpected agreement occurs, but also to identify when the calculations of emission rates, say, actually differ. This information may be just as valuable in elucidating the black hole/D-brane correspondence.

The spectrum of energy emitted in Hawking radiation [19] by a charged black hole is given by

$$\frac{dE}{dt}(\omega, q) = \frac{\omega |\mathcal{A}(\omega, q)|^2}{2\pi (e^{(\omega - q\Phi_e)/T_H} - 1)}, \quad (1)$$

where ω and q are the energy and charge of the emitted particle, T_H is the Hawking temperature and the chemical potential Φ_e is the difference in the electrostatic potential between infinity and the black hole horizon, $\Phi_e = \Phi_\infty - \Phi_h$. The prefactor $|\mathcal{A}(\omega, q)|^2$, required by detailed balance, is the classical absorption coefficient for the emitted mode.

In the case of neutral emission, $q = 0$ in (1), Das and Mathur [5] showed agreement between the D-brane and spacetime calculations of the absorption coefficient $\mathcal{A}(\omega) \equiv \mathcal{A}(\omega, q = 0)$ at leading order in a small ω approximation. Maldacena and Strominger [7] then derived a more detailed result for the spacetime calculation of $\mathcal{A}(\omega)$, showing striking detailed agreement with the D-brane result of [5] (though still in a low energy approximation). Below we show that a result for the spacetime absorption coefficient $\mathcal{A}(\omega)$ of the form given in [7] continues to hold over a larger region of black hole parameter space, yielding an expression which is invariant under U-duality transformations. The expression is symmetric under interchange of the three charges carried by the black hole, as one should expect for neutral emission.

This new expression for $\mathcal{A}(\omega)$ disagrees in general with the D-brane result. We then ask the question, given that calculations in the D-brane system are limited to processes in which the dynamics of branes and anti-branes are unimportant, over what region of parameter space should we expect the D-brane result to be accurate? We show that if one assumes that the statistical D-brane system is described by the U-duality invariant entropy formula given in [20], as it must be to correspond to the black hole, then the region of parameter space, for which left and right moving excitations dominate the process of energy exchange of the system with its environment, matches precisely the regime for which the D-brane and spacetime results for $\mathcal{A}(\omega)$ agree. This demonstrates a degree of consistency in the assumption that a full understanding of D-brane dynamics would reproduce the U-duality invariant results. The full expressions, moreover, may give clues towards such an understanding. For example, in the U-duality related limit in which 1-branes and anti-1-branes dominate, the expression for $\mathcal{A}(\omega)$ may be interpreted in terms of distributions of branes and anti-branes.

Turning to the case of charged emission, $q \neq 0$ in (1), contrary to the conclusions of [21, 7], we find real disagreement between the spacetime and D-brane results for $\mathcal{A}(\omega, q)$ in

an interesting region of parameter space. For charged emission, the spacetime result was first calculated and compared with the D-brane result [21,22] to leading order in the small parameter

$$\omega^2 - q^2. \tag{2}$$

A more detailed result for the spacetime absorption coefficient was given in [7], again showing apparent striking agreement with the D-brane result. We show here that this agreement does not extend over the entire parameter range of interest. In particular, it does not hold in the ‘fat black hole’ (hereafter FBH) regime [17,18]. We find that there are two cases to consider; Case I, in which the difference (2) is small, with both ω and q individually small; and Case II, in which the difference (2) is small, but neither ω nor q is small. We find that the D-brane and spacetime results for $\mathcal{A}(\omega, q)$ agree only in case I. Case II, however, applies in an interesting region of parameter space, the FBH regime. It is left as an open question, whether the lack of agreement we find between the spacetime and D-brane results in this case shows a real disagreement or an error in the D-brane result. Such an error could arise in the FBH regime because charge is stored on the brane in smaller units than may be carried by excitations in the bulk of spacetime [17][18]. Only very special combinations of brane excitations can actually annihilate to carry charge off the brane via the simple interactions considered in [5].

Thirdly, it should be recognized that the emission rate which has been computed for the D-brane system does not by any means hold over the entire black body spectrum, for either neutral or charged emission. In particular, as we will discuss below, in the high frequency limit the black hole absorption coefficient approaches unity. This is just the particle limit - an incident, zero angular momentum particle is absorbed by the black hole with unit probability. For large black holes, the characteristic frequency ω_{over} at which the absorption coefficient goes to one, though well above the Hawking temperature, is still small compared to the string scale. So at least naively, one should be able to use the effective vertex for weakly interacting left and right movers given in [6,5] in this regime.

This paper is organized as follows: In section 2 we review the properties of the family of spacetimes and the D-brane model, including the D-brane result for the absorption coefficient. In section 3, we establish, via thermodynamic arguments based on the self-consistency of the D-brane model, the regime in which we should expect agreement between the D-brane and spacetime results. Section 4 presents the calculation of the spacetime absorption coefficient. Section 5 is a comparison between the spacetime and D-brane results. Section 6 contains a discussion of the high energy limit of the absorption coefficient. We present our conclusions in section 7.

2. Black Holes and D-branes

2.1. The Spacetimes

The 5-dimensional black hole solutions of interest [20,23] are obtained from 10-dimensional, boosted, black brane solutions to the low energy effective action of Type IIB string theory, which contains the terms (in the 10d Einstein frame)

$$\frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left(R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{12}e^\phi H^2 \right), \quad (3)$$

where H is the RR 3-form, ϕ is the dilaton, $G_{10} = 8\pi^6 g^2$, $\alpha' = 1$ and g is the $d = 10$ string coupling. The $d = 10$ solutions are given by

$$\begin{aligned} ds^2 = & \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-3/4} \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{-1/4} [-dt^2 + dx_5^2 \\ & + \frac{r_0^2}{r^2}(\cosh \sigma dt - \sinh \sigma dx_5)^2 + \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right) \sum_{i=6}^9 dx_i^2] \\ & + \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{1/4} \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right)^{3/4} \left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right] \\ e^{-2\phi} = & \left(1 + \frac{r_0^2 \sinh^2 \gamma}{r^2}\right) \left(1 + \frac{r_0^2 \sinh^2 \alpha}{r^2}\right)^{-1} \\ H = & 2r_0^2 \sinh^2 \gamma \epsilon_3 + 2r_0^2 \sinh^2 \alpha e^{-\phi} *_6 \epsilon_3, \end{aligned} \quad (4)$$

where $*_6$ is the Hodge dual in the six dimensions t, \dots, x_5 and ϵ_3 is the volume element on the 3-sphere. The x_5 coordinate is taken to be compact with period $2\pi R$. The coordinates x_6, \dots, x_9 are also taken to be compact and are each identified with period $2\pi V^{1/4}$. The $d = 10$ solution is then specified by the six parameters $\alpha, \gamma, \sigma, r_0, R, V$. These may be exchanged for a set of six physical parameters; three charges Q_1, Q_5, n , the ADM energy E , R and V . The three charges are given by

$$\begin{aligned} Q_1 &= \frac{V}{4\pi^2 g} \int e^\phi *_6 H = \frac{V r_0^2}{2g} \sinh 2\alpha, \\ Q_5 &= \frac{1}{4\pi^2 g} \int H = \frac{r_0^2}{2g} \sinh 2\gamma, \\ n &= \frac{R^2 V r_0^2}{2g^2} \sinh 2\sigma. \end{aligned} \quad (5)$$

The third charge n is related to the momentum around the circle in the x_5 direction by $P = n/R$. The ADM energy is

$$E = \frac{RVr_0^2}{2g^2}(\cosh 2\alpha + \cosh 2\gamma + \cosh 2\sigma). \quad (6)$$

The black hole entropy S , Hawking temperature T_H and chemical potential μ_n (related by a factor $1/R$ to Φ_e above) are given by

$$S = \frac{2\pi RVr_0^3}{g^2} \cosh \alpha \cosh \gamma \cosh \sigma, \quad \frac{1}{T_H} = 2\pi r_0 \cosh \alpha \cosh \gamma \cosh \sigma, \quad \mu_n = \frac{\tanh \sigma}{R}. \quad (7)$$

The reduced five-dimensional metric and additional Kaluza-Klein moduli can be read off from

$$ds_{10}^2 = e^{2\chi} \sum_i dx_i^2 + e^{2\psi} (dx_5 + A_\mu dx^\mu)^2 + e^{-2(4\chi+\psi)/3} ds_5^2 \quad (8)$$

where $\mu = 0, 1, \dots, 4$. Using these normalizations, ds_5^2 is the Einstein metric in five dimensions, which for (4) takes the simple form

$$ds_5^2 = -f^{-2/3} \left(1 - \frac{r_0^2}{r^2}\right) dt^2 + f^{1/3} \left[\left(1 - \frac{r_0^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega_3^2 \right], \quad (9)$$

$$f = \left(1 + \frac{r_1^2}{r^2}\right) \left(1 + \frac{r_5^2}{r^2}\right) \left(1 + \frac{r_n^2}{r^2}\right),$$

where the characteristic radii $r_1 = r_0 \sinh \alpha$, $r_5 = r_0 \sinh \gamma$ and $r_n = r_0 \sinh \sigma$ are sometimes referred to as ‘charges’ below.

2.2. The D-brane Model

In reference [20] the six physical parameters above were further exchanged for a set of six ‘brane numbers’ via the relations

$$\begin{aligned} Q_1 &= N_1 - N_{\bar{1}}, & Q_5 &= N_5 - N_{\bar{5}}, & n &= N_L - N_R, \\ E &= \frac{R}{g}(N_1 + N_{\bar{1}}) + \frac{RV}{g}(N_5 + N_{\bar{5}}) + \frac{1}{R}(N_R + N_L), \\ V &= \left(\frac{N_1 N_{\bar{1}}}{N_5 N_{\bar{5}}}\right)^{1/2}, & R &= \left(\frac{g^2 N_R N_L}{N_1 N_{\bar{1}}}\right)^{1/4}. \end{aligned} \quad (10)$$

N_1 and $N_{\bar{1}}$ are thought of as the numbers of 1D-branes and anti-1D-branes wrapping around the circle in the x_5 -direction. Likewise, N_5 and $N_{\bar{5}}$ are thought of as the numbers

of 5D-branes and anti-5D-branes wrapping around the internal 5-torus, and N_L/R and N_R/R are the total momenta carried by massless, left and right moving, open string excitations propagating along the common x_5 coordinate of the branes. In terms of the boost parameters, α, γ, σ appearing in the metric (4), the brane numbers are given by

$$\begin{aligned} N_1 &= \frac{Vr_0^2}{4g}e^{2\alpha}, & N_{\bar{1}} &= \frac{Vr_0^2}{4g}e^{-2\alpha}, \\ N_5 &= \frac{r_0^2}{4g}e^{2\gamma}, & N_{\bar{5}} &= \frac{r_0^2}{4g}e^{-2\gamma}, \\ N_L &= \frac{r_0^2 R^2 V}{4g^2}e^{2\sigma}, & N_R &= \frac{r_0^2 R^2 V}{4g^2}e^{-2\sigma}. \end{aligned} \quad (11)$$

The black hole entropy, expressed in terms of the brane numbers, takes the simple form [20]

$$S = 2\pi \left(\sqrt{N_1} + \sqrt{N_{\bar{1}}} \right) \left(\sqrt{N_5} + \sqrt{N_{\bar{5}}} \right) \left(\sqrt{N_R} + \sqrt{N_L} \right). \quad (12)$$

At this level, the brane numbers represent only a relabeling of parameters. However, in the near extremal, weak string coupling limit, a string theory based counting argument, in which the brane numbers stand for actual numbers of branes, reproduces the appropriate limit of (12) as the statistical degeneracy of the system [3,24,20]. Very briefly, the construction is the following. If the charges Q_1, Q_5, n are all taken to be positive, then the extremal limit is $N_{\bar{1}} = N_{\bar{5}} = N_R = 0$. Moving slightly away from extremality, the expression for the energy E in (10) indicates that, if $R^2/g, R^2V/g \gg 1$, momentum modes will be light compared to anti-branes. Adding a small amount of energy will then cause N_R to increase, with $N_{\bar{1}}, N_{\bar{5}}$ remaining approximately zero. The counting arguments then proceed by counting the number of distinct configurations of left and right moving open string excitations having the correct total momentum, N_L and N_R .

In order correctly to reproduce the limiting form of the entropy (12) in the extremal limit, $N_{\bar{1}} = N_{\bar{5}} = N_L = 0$, it is necessary to assume that the momenta of open string excitations propagating along the brane are quantized in units of $1/L$, with $L = N_1 N_5 R$, rather than in units of $1/R$ as for closed string states propagating in the bulk of the spacetime [17,18]. We refer to this as the ‘fat black hole’ (hereafter FBH) prescription. How the FBH prescription arises from string theory has been explored recently in [25,26].

Moving away from extremality by adding right-movers, keeping $N_{\bar{1}}, N_{\bar{5}} \simeq 0$, in the limit $N_L, N_R \gg 1$, the microcanonical distributions of massless, left and right moving, open string excitations may be replaced by canonical distributions,

$$\rho_L(\omega_L) = \frac{1}{e^{\omega_L/T_L} - 1}, \quad \rho_R(\omega_R) = \frac{1}{e^{\omega_R/T_R} - 1}, \quad (13)$$

The temperatures T_L, T_R , determined by the conditions that the average total momenta carried by the left and right movers be $N_L/R, N_R/R$, are given by

$$\begin{aligned}\frac{1}{T_L} &= \pi R \sqrt{\frac{N_1 N_5}{N_L}} = \frac{\pi r_1 r_5}{2r_0^2} \left(\sqrt{r_n^2 + r_0^2} - r_n \right), \\ \frac{1}{T_R} &= \pi R \sqrt{\frac{N_1 N_5}{N_R}} = \frac{\pi r_1 r_5}{2r_0^2} \left(\sqrt{r_n^2 + r_0^2} + r_n \right).\end{aligned}\tag{14}$$

T_R vanishes in the extremal limit. The entropy (12) with $N_1 = N_5 = 0$ is then reproduced by the sum of two non-interacting one dimensional ideal gasses,

$$S = S_L + S_R = 2\pi^2(T_L + T_R)L.\tag{15}$$

The excited D-brane system, *i.e.* with $N_R > 0$, decays via closed string emission as described in [3]. A left moving open string having energy ω_L annihilates with a right moving open string having energy ω_R to form a closed string having energy $k_0 = \omega_L + \omega_R$ and internal momentum $k_5 = \omega_L - \omega_R$. From a 5-dimensional point of view, the closed string state carries electric charge $q = k_5$. Recall that the internal momentum k_5 of a closed string state is quantized in units of $1/R$, whereas the ω_L, ω_R are quantized in units of $1/L = 1/N_1 N_5 R$. This implies a strong restriction on which left and right movers may annihilate to form a closed string which can propagate into the bulk spacetime via this interaction.

The basic interaction vertex for the above process has been determined both from the low energy limit of perturbative string calculations [6] and from the Born-Infeld effective action for the D-brane system [5]. The rate for both neutral and charged emission can be calculated using the methods of [5] giving the result [21,22]

$$\frac{dE}{dt}(\omega, k_5) = \omega G_5 \frac{N_1 N_5 R}{2} (\omega^2 - k_5^2)^2 \rho_L(\omega_L) \rho_R(\omega_R),\tag{16}$$

where $G_5 = \pi g^2/4RV$. Comparing this expression with (1), the D-brane prediction for the absorption coefficient is then

$$|\mathcal{A}_D(\omega, k_5)|^2 = \pi G_5 N_1 N_5 R (\omega^2 - k_5^2) \frac{e^{(\omega - R\mu_n k_5)/T_H} - 1}{(e^{(\omega + k_5)/2T_L} - 1)(e^{(\omega - k_5)/2T_R} - 1)}\tag{17}$$

3. Expected Regime of Agreement Between D-brane and Spacetime Results

3.1. Neutral Emission

In section (4) we will obtain a result for the spacetime absorption coefficient $|\mathcal{A}(\omega, k_5)|^2$ which disagrees in general with the D-brane result $|\mathcal{A}_D(\omega, k_5)|^2$ in (17) above. We would

like to ask, in advance, given that the D-brane calculation does not take into account brane/anti-brane dynamics, over what regime of black hole parameter space should we expect the D-brane and spacetime results to agree?

For neutral emission ($k_5 = 0$), the D-brane system is exchanging a small amount of energy with its environment, the other physical parameters being held fixed. If the D-brane system does indeed correspond to the black hole, then we can use (10) to calculate how the brane numbers change under a small change in the energy E , with the other physical parameters Q_1, Q_5, n, R and V held fixed. We find these partial derivatives of the brane numbers to be given by

$$\begin{aligned}\frac{\partial N_1}{\partial E} &= \frac{\partial N_{\bar{1}}}{\partial E} = \frac{g \cosh 2\gamma \cosh 2\sigma}{2R\Omega} \\ \frac{\partial N_5}{\partial E} &= \frac{\partial N_{\bar{5}}}{\partial E} = \frac{g \cosh 2\alpha \cosh 2\sigma}{2RV\Omega} \\ \frac{\partial N_R}{\partial E} &= \frac{\partial N_L}{\partial E} = \frac{R \cosh 2\alpha \cosh 2\gamma}{2\Omega},\end{aligned}\tag{18}$$

where

$$\Omega = \cosh 2\alpha \cosh 2\gamma + \cosh 2\alpha \cosh 2\sigma + \cosh 2\gamma \cosh 2\sigma.\tag{19}$$

The dependence on g, R, V in these expressions reflects the masses of the different brane species, as discussed above. The dependence on the boost parameters α, γ, σ , arises from holding R, V fixed. We see that for the case of equal charges, $\alpha = \gamma = \sigma$, the change in a given brane number is simply the inverse of its mass per unit excitation.

The partial derivatives (18) are ingredients in a calculation of the Hawking temperature in terms of the brane numbers. We can write

$$\frac{1}{T_H} = \frac{\partial S}{\partial E} = \sum_i \left(\frac{\partial N_i}{\partial E} \right) \frac{\partial S}{\partial N_i}\tag{20}$$

where the index i runs over the six types of excitations. The derivatives $\partial S/\partial N_i$ are easily calculated from (12) and we arrive at

$$\frac{1}{T_H} = \frac{2\pi r_0 \cosh \alpha \cosh \gamma \cosh \sigma}{\Omega} \{ \cosh 2\gamma \cosh 2\sigma + \cosh 2\alpha \cosh 2\sigma + \cosh 2\alpha \cosh 2\gamma \},\tag{21}$$

which correctly reproduces the Hawking temperature (7) of the black hole, since the three terms in brackets sum to Ω . The three individual terms in (21) come respectively from 1-branes & anti-1-branes, 5-branes & anti-5-branes and left & right moving excitations. We see that the relative contributions have the same dependence on the boost parameters

as appear in (18). However, the factors of g, R, V arising from the masses of the different excitations have been normalized away by the coefficients $\partial S/\partial N_i$.

We suggest that it is the relative size of the contribution of each brane species to the Hawking temperature, as in (21), which determines its importance to processes of energy exchange, rather than the change in brane number itself. For example, with equal charges, $\alpha = \gamma = \sigma$, all three sets of excitations contribute equally to the temperature and are therefore equally important thermodynamically. By this criterion, we see from (21), using the expression for Ω (19), that left and right moving excitations dominate the expression for T_H in the limit $\sigma \ll \alpha, \gamma$, which is equivalent to the condition

$$r_n \ll r_1, r_5. \quad (22)$$

If this criterion is correct, it is in this regime that we should expect agreement between the D-brane and spacetime results. Note that the region of parameter space satisfying (22) is the same as the ‘dilute gas’ limit in reference [7]. In [7], equation (22) arose as the consistency condition for a small amplitude approximation of the dynamics of classical waves on a string. Here we see that the same condition emerges from considerations specific to the D-brane system under study. Moreover, in this way of deriving the condition (22), it is explicit that in the limit (22) the inverse Hawking temperature reduces as in [7] to

$$\frac{1}{T_H} = \frac{1}{2} \left(\frac{1}{T_L} + \frac{1}{T_R} \right). \quad (23)$$

It is also clear from the above analysis that in the regimes related by U-duality to (22), *i.e.* $r_1 \ll r_n, r_5$ and $r_5 \ll r_1, r_n$, the dynamics will be dominated by 1-branes and 5-branes respectively. In the limit

$$r_1 \ll r_n, r_5, \quad (24)$$

for example, the Hawking temperature is approximately

$$\frac{1}{T_H} = \frac{1}{2} \left(\frac{1}{T_1} + \frac{1}{T_{\bar{1}}} \right), \quad (25)$$

where

$$\frac{1}{T_1} = \frac{2\pi r_5 r_n}{r_0^2} \left(\sqrt{r_0^2 + r_1^2} - r_1 \right), \quad \frac{1}{T_{\bar{1}}} = \frac{2\pi r_5 r_n}{r_0^2} \left(\sqrt{r_0^2 + r_1^2} + r_1 \right) \quad (26)$$

can be thought of as the temperatures of canonical distributions of 1-branes and anti-1-branes. These same temperatures will arise in our expression for the spacetime absorption below after taking the limit (24).

3.2. Charged Emission

Our result for the spacetime absorption coefficient $|\mathcal{A}(\omega, k_5)|^2$ with $k_5 \neq 0$ disagrees as well in general with the D-brane result (17). So, again we would like to ask over what region of parameter space we should expect agreement. We can perform an analysis similar to that above, but now for a process in which the energy and charge change in a fixed ratio. To understand which excitations dominate the emission of charge, we calculate the chemical potential μ_n , for emitting particles carrying n -charge in terms of the brane numbers,

$$\frac{\mu_n}{T} = -\frac{\partial S}{\partial n} = -\sum_i \left(\frac{\partial N_i}{\partial n} \right) \left(\frac{\partial S}{\partial N_i} \right), \quad (27)$$

where the partial derivatives are taken holding E, Q_1, Q_5, R and V constant. We find the derivatives of the brane numbers are

$$\begin{aligned} \frac{\partial N_R}{\partial n} &= \frac{1}{2\Omega} (\cosh 2\alpha \cosh 2\gamma + \cosh 2\alpha e^{2\sigma} + \cosh 2\gamma e^{2\sigma}), \\ \frac{\partial N_L}{\partial n} &= \frac{1}{2\Omega} (\cosh 2\alpha \cosh 2\gamma + \cosh 2\alpha e^{-2\sigma} + \cosh 2\gamma e^{-2\sigma}), \\ \frac{\partial N_1}{\partial n} &= \frac{\partial N_{\bar{1}}}{\partial n} = -\frac{g \cosh 2\gamma \sinh 2\sigma}{2R^2\Omega}, \\ \frac{\partial N_5}{\partial n} &= \frac{\partial N_{\bar{5}}}{\partial n} = -\frac{g \cosh 2\alpha \sinh 2\sigma}{2R^2\Omega}, \end{aligned} \quad (28)$$

Assembling these into a calculation of the chemical potential μ_n gives

$$\begin{aligned} \frac{\mu_n}{T_H} &= \frac{2\pi r_0 \cosh \alpha \cosh \gamma \cosh \sigma}{\Omega} \left\{ 2 \cosh 2\gamma \cosh^2 \sigma + 2 \cosh 2\alpha \cosh^2 \sigma \right. \\ &\quad \left. - \cosh 2\gamma - 2 \cosh 2\alpha + \cosh 2\alpha \cosh 2\gamma \right\} \frac{\tanh \sigma}{R}. \end{aligned} \quad (29)$$

The terms in brackets sum to Ω giving the correct result $\mu_n = \tanh \sigma / R$ as in (7). In (29) the first two terms come from 1-branes & anti-1-branes and 5-branes & anti-5-branes respectively. The remaining terms all come from right & left moving excitations. Again, in the limit $\sigma \ll \alpha, \gamma$, corresponding to the limit (22), we see that the contributions of right/left moving excitations dominate. If we now consider a process in which the energy changes by k_0 and the charge n by Rk_5 , we can see that the most important processes will be those involving right/left movers in the regime (22). So, again we should expect to find agreement between D-brane and spacetime results in the regime $r_n \ll r_1, r_5$.

4. Calculating the Spacetime Absorption Coefficient

4.1. The Wave Equation

We now turn to the calculation of the spacetime absorption coefficient. Consider a massless scalar field Φ in ten dimensions, minimally coupled to the ten dimensional metric. The field is taken to be spherically symmetric, have frequency ω , momentum k_5 in the x_5 and to be independent of the other compact directions. From a five dimensional point of view, the field will then carry charge $q = k_5$. Let

$$\Phi = e^{-i\omega t} e^{ik_5 x_5} \varphi(r) \quad (30)$$

where $k = m/R$ with m an integer. The ten dimensional Klein-Gordon equation becomes

$$\frac{\left(1 - \frac{r_0^2}{r^2}\right)}{r^3} \partial_r \left[r^3 \left(1 - \frac{r_0^2}{r^2}\right) \partial_r \varphi \right] + \left[\omega_\infty^2 + \frac{r_n^2}{r^2} \mu^2 \right] \left(1 + \frac{r_1^2}{r^2}\right) \left(1 + \frac{r_5^2}{r^2}\right) f(r) \varphi = 0 \quad (31)$$

where $f(r)$ is as in (9), $\omega_\infty^2 = \omega^2 - k_5^2$ is the frequency squared of the wave at spatial infinity and $\mu = \omega - k_5(1 + r_0^2/r_n^2)^{\frac{1}{2}}$. In the extremal limit $r_0 = 0$, μ is the frequency of the wave near the horizon.

Transforming to the new radial coordinate $v = \frac{r_0^2}{r^2}$, the wave equation (31) becomes

$$(1-v) \frac{d}{dv} ((1-v) \phi'(v)) + \left[D + \frac{C}{v} + \frac{C_2}{v^2} + \frac{C_3}{v^3} \right] \phi = 0 \quad (32)$$

where the constant coefficients C, C_1, C_2 and D are given by

$$\begin{aligned} C_2 &= \frac{1}{4} (\omega_\infty^2 (r_1^2 + r_5^2) + \mu^2 r_n^2), & C_3 &= \frac{1}{4} r_0^2 \omega_\infty^2 \\ C &= \frac{1}{4r_0^2} (\omega_\infty^2 r_1^2 r_5^2 + \mu^2 (r_1^2 r_n^2 + r_5^2 r_n^2)), & D &= \frac{\mu^2}{4r_0^4} r_1^2 r_5^2 r_n^2, \end{aligned} \quad (33)$$

For vanishing internal momentum k_5 , $\mu = \omega_\infty$ and all four coefficients are symmetric under interchange of r_1, r_5 and r_n .

The scattering problem we consider below is of the same form as that considered by Maldacena and Strominger [7]. However, our treatment will differ from that in [7] in two important respects. First, we will not make the restriction $r_n \ll r_1, r_5$. This turns out to be unnecessary, and, consequently, we arrive at a result for the spacetime absorption coefficient valid over a larger, U-duality invariant region of parameter space. Secondly, we treat the matching problem involved in calculating the absorption coefficient more

carefully, extracting limits on the range of parameters over which the matching is a good approximation.

Specifically, we find that the spacetime absorption coefficient for general choices of r_1, r_5, r_n continues to have the form found in [7]

$$|\mathcal{A}(\omega, k_5)|^2 = \omega_\infty^2 r_0^2 \pi^2 ab \frac{e^{2\pi(a+b)} - 1}{(e^{2\pi a} - 1)(e^{2\pi b} - 1)}, \quad (34)$$

where

$$a = \sqrt{C+D} + \sqrt{D}, \quad b = \sqrt{C+D} - \sqrt{D} \quad (35)$$

and the coefficients C and D are as in (33). This result is valid with the frequency at infinity bounded by the condition.

$$\omega_\infty r_{max} \leq \frac{r_0}{r_{max}} \ll 1, \quad (36)$$

where r_{max} is the largest of r_1, r_5, r_n and r_0 is restricted by $r_0^3 \ll r_1 r_5 r_n$. Note that $\omega_\infty = \sqrt{(\omega - k_5)(\omega + k_5)}$ may be small either because both ω and k_5 are small, or because one of the factors $\omega \pm k_5$ is small. This will be important later on.

Two different kinds of expansion techniques have been used recently to calculate the absorption coefficient. One approach, developed in earlier work by Unruh [27] and Page [28], and applied in the present context in [4,5,21,22] is to note that the wave equation (31) is exactly solvable for $\omega_\infty = \mu = 0$ (equivalently, $\omega = k_5 = 0$). The solution $\phi(\omega_\infty, \mu; r)$ with finite parameters can then be expanded in a double power series in ω_∞ and μ . However, this expansion is valid only in a “middle region”, neither too close to the horizon, nor to infinity. This is because the zeroth order solution is singular on the horizon, and does not approach a plane wave at infinity. One then properly works with three regions. In the tortoise coordinate, the solutions are asymptotically Bessel functions near infinity and plane waves near the horizon. These pieces can be linked together by matching with the low frequency expansion in between. The result for the absorption coefficient is then given in terms of a power series expansion for small ω_∞, μ .

A second type of expansion, which yields results beyond a power series expansion for the absorption coefficient, was developed for rotating and charged four dimensional black holes in [29,30,31]. In these cases the wave equation can be solved asymptotically both near the horizon and near infinity in terms of special functions (hypergeometric and confluent hypergeometric functions), **and** there is an overlap region in which the asymptotic forms of

the solutions can be matched. The existence of this overlap region requires some parameter to be small, for example $\omega - m\Omega_H$ in the rotating case, where Ω_H is the angular velocity of the horizon and m is the angular momentum of the scattered wave. A similar approach the present context was introduced in [7] to obtain results of the form (34). In this case the solutions near infinity and near the horizon are Bessel functions and hypergeometric functions, respectively. However, in this case, as we will see below, there turns out to be no overlap region in which one can match the two asymptotic regimes. Therefore one needs again to introduce a “middle region”, and match the three parts of the solution, which we do below.

4.2. Near Infinity

The wave equation (32) becomes Bessel’s equation in the regime where the terms with coefficients C and D may be dropped relative to those with coefficients C_2, C_3 . Such an approximation is valid for $r \gg r_{max}$, where again r_{max} is the largest of r_1, r_5 and r_n . Normalizing the incoming part of the wave to unit amplitude, the solution ϕ_∞ near infinity is given in terms of the r coordinate by

$$\phi_\infty(r) = \sqrt{\frac{\omega_\infty \pi}{2}} \frac{e^{-i\pi/4}}{r} \left(H_\nu^{(2)}(\omega_\infty r) + i\mathcal{S} H_\nu^{(1)}(\omega_\infty r) \right), \quad \nu = 1 - \omega_\infty^2(r_1^2 + r_5^2 + r_0^2) + \mu^2 r_n^2. \quad (37)$$

Here \mathcal{S} is the scattering coefficient, related to the absorption coefficient $|\mathcal{A}|^2$ by the unitarity relation $|\mathcal{S}|^2 + |\mathcal{A}|^2 = 1$. The goal of the calculation then is to determine \mathcal{S} given the boundary condition that at the horizon the outgoing part of the wave vanishes. Equation (37) is a large r (small v) solution, however the Bessel functions may still be expanded for small argument, provided

$$\omega_\infty r_{max} \ll 1 \quad (38)$$

In terms of v , this expansion yields the leading terms

$$\phi_\infty \sim b_1 v + b_2 + b_3 \log v, \quad v \gg \omega_\infty^2 r_0^2 \quad (39)$$

where the coefficients b_i depend on \mathcal{S} and ω_∞ and are given in the appendix.

4.3. Near the Horizon

The near horizon regime is defined by keeping the C and D terms in the wave equation (32) and dropping the C_2 and C_3 terms. Several conditions are required for this approximation to be valid and different limiting cases are important. One necessary condition in all cases is

$$r_0^3 \ll r_1 r_5 r_n. \quad (40)$$

Over the range $r_1 \sim r_5$, $r_n \leq r_1$ (which includes both the equal charge case and $r_n \ll r_1$), a second condition deduced from (32), (33) is $r \ll \sqrt{r_n r_1}$. The solution ϕ_h in this regime, with the boundary condition that there is no outgoing flux at the horizon is the hypergeometric function [7]

$$\phi_h = A(1-v)^{-i(a+b)/2} F(-ia, -ib, 1-ia-ib, 1-v), \quad (41)$$

where the constants a and b have been defined in terms of the parameters of the wave equation in (35), and the constant A is to be determined through the matching procedure. This has the expansion as $v \rightarrow 1$

$$\phi_h \sim A \exp\{-i\sqrt{C+D} \log(1-v)\} \quad , \quad v \rightarrow 1. \quad (42)$$

Expanding ϕ_h for $v \rightarrow 0$, away from the horizon, one finds the leading terms

$$\phi_h \sim AE(1 + gv - abv \log v). \quad (43)$$

Here the function $E(a, b)$ was defined in [7] and is given in (A.4) the appendix below, and the constant g is given by

$$g = \frac{i}{2}(a+b) + ab(1 - 2\gamma - \psi(1-ia) - \psi(1-ib)), \quad (44)$$

where ψ is the digamma function and $\gamma = -\psi(1)$ is Euler's constant. For the expansion (43) to be valid, with the terms ordered as written (which will prove to be important in the matching below), v must satisfy

$$abv |\log v| \ll |g|v \ll 1. \quad (45)$$

We find by plugging in the actual coefficients a, b, g that these inequalities require $v \gg \Delta$, where $\Delta \approx e^{-\pi}$, and also imply a restriction on the frequency

$$\omega_\infty r_{max} \leq \frac{r_o}{r_{max}}. \quad (46)$$

Since r_0/r_{max} is itself constrained to be small by (40), equation (46) tells us how small ω_∞ needs to be for our calculation to hold. In the notation of [7], $\omega_\infty r_1 \sim O(\frac{r_o}{r_1})$ corresponds to $ab \sim O(1)$. Equation (46) is the condition for $k_5 \geq 0$, *i.e.* charge equal to zero, or the same sign as the black hole, which are the processes with the greatest emission rates. For $k_5 < 0$, the analogue of (46) is $\mu r_n \leq r_0/r_{max}$.

4.4. Matching

The result for the absorption coefficient in [7] comes from matching the constant and linear terms in the expansions of ϕ_∞ (39) and ϕ_h (43). It is important to note, however, that there is no actual interval in which the asymptotic forms ϕ_∞ and ϕ_h are both good approximate solutions. If we consider the equal charge case, $r_1 = r_5 = r_n$, and take $k_5 = 0$, then ϕ_∞ is a good approximation for $r \gg r_1$, while ϕ_h is valid for $r \ll r_1$, giving a clear gap. Moreover, if we consider the limit in which one of the charges is much smaller than the other two, such as $r_n \ll r_1, r_5$ with $r_1 \approx r_5$, then the gap widens further. The condition for ϕ_∞ remains $r \gg r_1$. However, the condition for ϕ_h deduced from (32) and (33) becomes $r \ll \sqrt{r_n r_1}$. Recall that this is the parameter range in which agreement is expected with the D-brane result. It is clear then, that, in general, there is a need for a third approximate solution in a middle region, which can be matched on to ϕ_∞ and ϕ_h at either end.

Unfortunately, we have been unable to find such an approximate solution and middle region with overlaps at both ends. However, an alternative way to see the need for a middle region is to note that the analytic forms for ϕ_h and ϕ_∞ in equations (39) and (43), which we want to match, disagree functionally¹. In particular, ϕ_h contains a $v \log v$ term for small v , while ϕ_∞ has a $\log v$ term, and a priori one expects the log's to be important for small v . In particular, the $v \log v$ term in ϕ_h is lower order than the v term as $v \rightarrow 0$. With this in mind, we try a series expansion of the form

$$\phi_{mid} = A_0 + A_1 v + \dots + \log v (B_0 + B_1 v + \dots), \quad (47)$$

which has the appropriate analytic structure to link together the analytic forms of ϕ_∞ and ϕ_h . Substituting ϕ_{mid} into the wave equation (32), we find that ϕ_{mid} is a good approximate solution for $r \ll r_{max}$, giving the coefficients B_n , $n \geq 0$ and A_n , $n \geq 2$ in terms of A_0 and A_1 . For example,

$$B_0 = C_2 A_0 + C_3 A_1, \quad B_1 = -\frac{C_2^2}{C_3} A_0 - C_2 A_1 \quad (48)$$

A_0 and A_1 are then two arbitrary constants determined by the matching. Matching the constant and linear terms between ϕ_h , ϕ_{mid} and ϕ_∞ then gives

$$b_2 = A_0 = AE, \quad b_1 = A_1 = AEg. \quad (49)$$

¹ For a discussion of matched asymptotic expansions see *e.g.* [32].

Matching the $\log v$ and $v \log v$ terms with those in ϕ_∞ and ϕ_h then gives nontrivial consistency checks on the solution, as there are no free parameters left. One can show that the coefficient of the $\log v$ term in ϕ_{mid} indeed matches the coefficient b_3 in the small argument expansion of the Bessel function ϕ_∞ (see appendix for details). This is good, since $\log v$ diverges as v approaches zero, and one would, therefore, expect this term to be important. However, we find that the coefficient B_1 of the $v \log v$ term in ϕ_{mid} does not match the corresponding coefficient in the expansion (43) of the hypergeometric function ϕ_h .

Does this lack of agreement pose a real problem? The regions of validity for ϕ_h and ϕ_{mid} overlap for

$$\Delta \ll v \ll 1 \quad (50)$$

If v is in this range, then $|abv \log v| \ll |gv|$, and it is reasonable to keep the linear term and ignore the logarithm in the expansion of the hypergeometric function. Still, we should take this as a cautionary note; the lack of matching implies that using the form of the hypergeometric function away from the horizon as given in (43) may not be valid. However, the expansion (43) is the key ingredient which yields the detailed form of $\mathcal{A}(\omega)$, which agrees so well with the D-brane prediction [7].

It is our view that this matching is correct in the parameter regimes indicated. As mentioned before, ϕ can be expanded in a double power series in ω_∞, μ in a middle region. This is a well defined expansion, and yields a form of ϕ_{mid} which is the same as in (47); the zeroth order term is $\phi \sim a_0 + a_1 v$ and the first order terms gives the logarithms. The virtue of the expansion is that it is well defined; the drawback is that from it one only obtains the very low frequency form of the absorption coefficient $|\mathcal{A}(\omega, k_5)|^2 \sim \omega \omega_\infty^2$.

Finally, let us complete the calculation of $A(\omega, k_5)$. The matching conditions (49) imply that $b_1/b_2 = g$, which allows us to solve for \mathcal{S} in terms of g , a known function of the parameters in the wave equation. This gives

$$1 + i\mathcal{S} = -i\frac{\pi}{2}\omega^2 r_0^2 g \quad (51)$$

Using properties of the digamma function given in the appendix and the relation $|\mathcal{A}|^2 = 1 - |\mathcal{S}|^2$, then gives the expression (34) for the absorption coefficient, completing the derivation.

4.5. Flux Computation

The conserved particle number current of the Klein-Gordon equation can also be used to compute the absorption coefficient, as in [27,7]. We use this method to generate a consistency check on our solution, since the current should be independent of radius r . In terms of the tortoise coordinate r_* , the flux is

$$\mathcal{F} = \frac{1}{2i} f^{1/2} r^3 (\phi^* \frac{\partial \phi}{\partial r_*} - c.c.) \quad (52)$$

Using ϕ_∞ , (37), this gives as $r \rightarrow \infty$, a sum of the incident flux and the outgoing scattered flux,

$$\mathcal{F} = \omega_\infty (-1 + |\mathcal{S}|^2) \quad (53)$$

On the other hand, using the expression (42) for ϕ near the horizon, the flux is

$$\begin{aligned} \mathcal{F} &= \frac{1}{2i} (2r_0^2 (1-v) \phi^* \partial_v \phi - c.c.) \\ &= -r_0^2 (a+b) |A|^2, \quad v \rightarrow 1 \end{aligned} \quad (54)$$

Now, if one uses the expansion of the hypergeometric function away from the horizon, equation (43), the flux must be the same. Again, using (A.6) this works out,

$$\begin{aligned} \mathcal{F} &= -ir_0^2 (a+b) |AE|^2 (g - g^*), \quad v \rightarrow 0 \\ &= -r_0^2 (a+b) |AE|^2 |E|^{-2} \end{aligned} \quad (55)$$

Further, one could use ϕ_{mid} or the small argument expansion of ϕ_∞ to compute the flux. For example, using either (47) or (39), the flux in the region $\omega^2 r_0^2 \ll v \ll 1$ is

$$\begin{aligned} \mathcal{F} &= -ir_0^2 (b_2^* b_1 - b_1^* b_2) \\ &= -ir_0^2 (a+b) |AE|^2 (g - g^*). \end{aligned} \quad (56)$$

Finally, the absorption coefficient is the ratio of the flux into the horizon to the incident flux at infinity. From (53) the latter is just ω_∞ , and we again get (34) for \mathcal{A} .

5. Agreement and Disagreement with D-Brane Prediction

We have now established that the spacetime absorption coefficient has the form (34), which we repeat here,

$$|\mathcal{A}(\omega, k_5)|^2 = \omega_\infty^2 r_0^2 \pi^2 ab \frac{e^{2\pi(a+b)} - 1}{(e^{2\pi a} - 1)(e^{2\pi b} - 1)}, \quad (57)$$

with constants a and b given by

$$\begin{aligned}
a &= \sqrt{C+D} + \sqrt{D}, & b &= \sqrt{C+D} - \sqrt{D} \\
C &= \frac{1}{4r_0^2} (\omega_\infty^2 r_1^2 r_5^2 + \mu^2 (r_1^2 r_n^2 + r_5^2 r_n^2)), & D &= \frac{\mu^2}{4r_0^4} r_1^2 r_5^2 r_n^2, \\
\omega_\infty^2 &= \omega^2 - k_5^2, & \mu &= \omega - k_5 (1 + r_0^2/r_n^2)^{\frac{1}{2}},
\end{aligned} \tag{58}$$

This result is valid under the conditions $r_0^3 \ll r_1 r_5 r_n$ and $\omega_\infty r_{max} \leq r_0/r_{max} \ll 1$. This range of parameters is invariant under U-duality transformations interchanging the three charges.

5.1. U-duality and Neutral Emission

For neutral emission ($k_5 = 0$), the absorption coefficient (57) is itself invariant under interchange of the three charges r_1, r_5, r_n and in general disagrees with the D-brane result for the absorption coefficient in (17). We saw in section (2), that we should expect agreement only over the range $r_n \ll r_1, r_5$. In [7] the spacetime calculation was done restricted to this parameter range, yielding remarkable agreement with the D-brane result. Here we simply show that the approximation $r_n/r_1, r_n/r_5 \ll 1$ can be made on our final result as well to get agreement with the D-brane result.

In the neutral case, the coefficients a, b in (57), (58) can be put in the form

$$\begin{aligned}
a &= \frac{\omega r_1 r_5 r_n}{2r_0^2} \left(\sqrt{1 + \frac{r_0^2}{r_n^2} + \frac{r_0^2}{r_1^2} + \frac{r_0^2}{r_5^2}} + 1 \right), \\
b &= \frac{\omega r_1 r_5 r_n}{2r_0^2} \left(\sqrt{1 + \frac{r_0^2}{r_n^2} + \frac{r_0^2}{r_1^2} + \frac{r_0^2}{r_5^2}} - 1 \right).
\end{aligned} \tag{59}$$

Making the approximations $r_0, r_n \ll r_1, r_5$ one recovers the result of [7]

$$\begin{aligned}
a &\simeq \frac{\omega r_1 r_5}{2r_0^2} \left(\sqrt{r_0^2 + r_n^2} + r_n \right) = \frac{\omega}{4\pi T_R} \\
b &\simeq \frac{\omega r_1 r_5}{2r_0^2} \left(\sqrt{r_0^2 + r_n^2} - r_n \right) = \frac{\omega}{4\pi T_L},
\end{aligned} \tag{60}$$

leading to agreement between the exponential factors in (57) and (17). The prefactors can also be seen to agree in this limit, implying that our expression for the spacetime absorption coefficient agrees overall in this limit with the D-brane result, as in [7].

If instead of $r_n \ll r_1, r_5$, we consider, for example, the U-duality related limit $r_1 \ll r_n, r_5$, then the constants a and b in (57) reduce to

$$\begin{aligned} a &\simeq \frac{\omega r_n r_5}{2r_0^2} \left(\sqrt{r_0^2 + r_1^2} + r_1 \right) = \frac{\omega}{4\pi T_1} \\ b &\simeq \frac{\omega r_n r_5}{2r_0^2} \left(\sqrt{r_0^2 + r_1^2} - r_1 \right) = \frac{\omega}{4\pi T_1}, \end{aligned} \quad (61)$$

where the temperatures $T_1, T_{\bar{1}}$ were defined above in (26). It is then tempting to think of the exponential factors in the denominator in (57) as arising from Bose-Einstein distributions of 1-branes and anti-1-branes. The analogue of the quantized momenta of the left and right-movers, might then be the number of times a given 1-brane, or anti-1-brane, is wound. Perhaps more careful analysis of (57) in this limit (or the similar limit for 5-branes) could yield insight into brane/anti-brane dynamics.

5.2. Charged emission

Analysis of the charged case is complicated by the difference between the factors ω_∞ and μ . Validity of the spacetime calculation requires $\omega_\infty r_{max}$ to be small compared to r_0/r_{max} . However, this does not imply that μ is necessarily small as well. We consider two cases. Case I, in which $\omega_\infty^2 = (\omega - k_5)(\omega + k_5)$ is small, with both ω and k_5 individually small; and Case II, in which ω_∞^2 is small because either the difference or sum of ω and k_5 is small, with neither one individually small. In Case I, $\mu = \omega - \sqrt{1 + r_0^2/r_n^2} k_5$ is also small, of the same order as ω_∞ . However, in Case II, μ will be small only if r_0/r_n is also small (and it is the difference of ω and k_5 , which is small, rather than the sum).

Let us see what effect this has on the coefficients a and b , which may be written as

$$\begin{aligned} a &= \frac{r_1 r_5 r_n}{2r_0^2} \left(\sqrt{\omega_\infty^2 \frac{r_0^2}{r_n^2} + \mu^2 \left(1 + \frac{r_0^2}{r_1^2} + \frac{r_0^2}{r_5^2} \right)} + \mu \right), \\ b &= \frac{r_1 r_5 r_n}{2r_0^2} \left(\sqrt{\omega_\infty^2 \frac{r_0^2}{r_n^2} + \mu^2 \left(1 + \frac{r_0^2}{r_1^2} + \frac{r_0^2}{r_5^2} \right)} - \mu \right). \end{aligned} \quad (62)$$

Case I: ω_∞ and μ are the same order of magnitude. In the limit $r_0, r_n \ll r_1, r_5$, the r_0/r_1 and r_0/r_5 terms inside the square root may then be consistently dropped relative to the other terms and we get

$$a \simeq \frac{\omega - k_5}{4\pi T_R}, \quad b \simeq \frac{\omega + k_5}{4\pi T_L}, \quad (63)$$

giving agreement between the exponential factors in (57) and (17), as in [7]. Once again the prefactors may also be seen to agree in this limit, giving overall agreement between the two expressions. As in the neutral case, the spacetime and D-brane results disagree for r_n outside of the limit $r_n \ll r_1, r_5$.

Case II: Assuming that r_0/r_n is not much smaller than 1, then μ is not small in this case. The $\omega_\infty^2 r_0^2/r_n^2$ term under the square root in a, b then can have the same order of magnitude as the $\mu^2 r_0^2/r_1^2$, $\mu^2 r_0^2/r_5^2$ terms, and the latter may not be consistently dropped relative to the former. The expressions for a, b then do not reduce to the form (63) necessary to obtain agreement with the D-brane result. With $r_0/r_n \ll 1$, it is simple to see that again the $\omega_\infty^2 r_0^2/r_n^2$ term may be of the same order as, or less than, the $\mu^2 r_0^2/r_1^2$, $\mu^2 r_0^2/r_5^2$ terms, giving disagreement with the D-brane result.

5.3. Charged Emission Via Boosts

Another way to analyse the results for charged emission is to exploit, as in [7], the boost invariance of the scattering problem. In [7] it was shown that the wave equation for a scalar, with energy ω and internal momentum k_5 , can be rewritten in terms of the neutral ($k_5 = 0$) equation, with boosted parameters given by

$$\omega'^2 = \omega^2 - k_5^2, \quad r'_n = r_0 \sinh \sigma', \quad e^{\pm \sigma'} = e^{\pm \sigma} \frac{(\omega \mp k_5)}{\omega'}. \quad (64)$$

In terms of the boosted parameters, the coefficients C and D are

$$C = \frac{\omega'^2}{4r_0^2} (r_1^2 r_5^2 + r_1^2 r_n'^2 + r_5^2 r_n'^2), \quad D = \frac{\omega'^2}{4r_0^4} r_1^2 r_5^2 r_n'^2. \quad (65)$$

The constants a and b in the absorption coefficient are then

$$a = \frac{\omega' r_1 r_5 r_n'}{2r_0^2} \left(\sqrt{\frac{r_0^2}{r_n'^2} + \frac{r_0^2}{r_5^2} + \frac{r_0^2}{r_1^2} + 1} + 1 \right), \quad (66)$$

$$b = \frac{\omega' r_1 r_5 r_n'}{2r_0^2} \left(\sqrt{\frac{r_0^2}{r_n'^2} + \frac{r_0^2}{r_5^2} + \frac{r_0^2}{r_1^2} + 1} - 1 \right).$$

In the limit

$$r_0, r_n' \ll r_1, r_5 \quad (67)$$

we recover the result [7],

$$\begin{aligned} a &\simeq \frac{\omega' r_1 r_5}{2r_0^2} \left(\sqrt{r_0^2 + r_n'^2} + r_n' \right) = \frac{\omega - k_5}{4\pi T_R} \\ b &\simeq \frac{\omega' r_1 r_5}{2r_0^2} \left(\sqrt{r_0^2 + r_n'^2} - r_n' \right) = \frac{\omega + k_5}{4\pi T_L}. \end{aligned} \quad (68)$$

We then have agreement with the D-brane case, so long as (67) holds. We can now ask for what range of the original unboosted parameters (67) fails. From the definition of the boosted parameters in (64), we have

$$e^{\sigma'} = \sqrt{\frac{\omega - k_5}{\omega + k_5}} e^{\sigma}, \quad e^{-\sigma'} = \sqrt{\frac{\omega + k_5}{\omega - k_5}} e^{-\sigma}. \quad (69)$$

From this we see that $r_n' = r_0 \sinh \sigma'$ can become large (for fixed r_0) in two different ways, either by $\sigma \rightarrow \pm\infty$, which would make r_n itself large, or by having $(\omega - k_5)/(\omega + k_5)$ tend to zero or infinity, keeping σ finite. This latter possibility in which (67) fails is exactly Case II above, in which the sum or difference of ω and k_5 is small, with neither individually small. We have then reproduced the results of the previous analysis in terms of the boosted parameters.

5.4. Disagreement in the Fat Black Hole Regime

In section (2) we argued that the D-brane and spacetime results for the absorption coefficient should agree over the parameter range $r_0, r_n \ll r_1, r_5$. We have found, however, that the results for charged emission actually disagree within this parameter range, when the quantity $\omega_\infty^2 = (\omega - k_5)(\omega + k_5)$ is small, without either ω or k_5 being small individually. We now show that this is characteristic of low energy, charged emission in the FBH regime.

Recall that in the FBH regime, open string excitations on the brane have ω , k_5 quantized as integer multiples of $1/N_1 N_5 R$. Whereas, modes which propagate in the bulk have k_5 quantized as an integer multiple of $1/R$. The lowest energy charged emission possible would be, *e.g.*, a left moving open string with energy and momentum $\omega_L = k_{5,L} = (N_1 N_5 + 1)/N_1 N_5 R$ and a right moving open string with $\omega_R = -k_{5,R} = 1/N_1 N_5 R$, annihilating to form a closed string with energy and internal momentum $\omega = (N_1 N_5 + 2)/N_1 N_5 R$, $k_5 = 1/R$. This process then falls, for $N_1 N_5 \gg 1$, within our Case II above, in which the difference between ω and k_5 is much smaller than the sum (or alternately, sending $r' \rightarrow \infty$), and the spacetime and D-brane results disagree.

It should be noted that the size of the disagreement we have found between the spacetime and D-brane results is not large². We have observed above that in the FBH limit, we can have $r'_n \rightarrow r_1, r_5$, while r_n itself is still small. The term $r_0^2/r_n'^2$ in (66) is then of the same order as the r_0^2/r_1^2 and r_0^2/r_5^2 . The latter terms then cannot be ignored relative to the former. The exponential terms in the D-brane and spacetime results then differ by terms of this order. As a check that our spacetime result is accurate to this order, it is straightforward to verify that the low frequency limit of (57) yields the black hole area, as the results of [33] demand, to this order³.

6. Disagreement in Higher Energy Scattering

The wave equation (31) can also be solved for the scattering coefficient \mathcal{S} at frequencies sufficiently high that the wave is over the scattering barrier. At these energies, one can use the Born approximation. We will see that the absorption coefficient rapidly goes to one - this is just the particle limit, in which the particle is captured. However, the D-brane prediction for the emission rate, still given by (16) and no longer matches the Hawking emission in this higher frequency part of the spectrum. For classical sized black holes, these higher frequencies are still well below the string scale, at which corrections to the approximate vertex [6,5] used in the computation of (17) should be important.

First rewrite the wave equation in a standard scattering form. The tortoise coordinate is defined by

$$dr_* = \frac{\sqrt{f}}{1 - \frac{r_0^2}{r^2}} dr \quad (70)$$

Let $\lambda = r^{3/2} f^{1/4}$ and let $\chi = \lambda \phi$. Then the wave equation (31) becomes

$$\chi''(r_*) + [\omega_\infty^2 - V_{coul} - V_{grav}] \chi = 0, \quad (71)$$

where

$$V_{coul} = r_n^2 \frac{\omega_\infty^2 - \mu^2}{r^2 + r_n^2}, \quad V_{grav} = \frac{\lambda''(r_*)}{\lambda}, \quad (72)$$

and prime denotes differentiation with respect to r_* . The total potential falls off like r_*^{-2} as $r_* \rightarrow \infty$, so the parameter ω_∞ is the frequency at infinity. The ‘‘Coulombic’’ potential

² We thank J. Maldacena for emphasizing this point to us.

³ An additional order of accuracy in r_0^2/r_1^2 can be obtained by considering the U-duality invariant extension of the results of [13]. See Note Added below.

is monotonic decreasing from the horizon to infinity. The “gravitational” potential falls off exponentially fast towards the horizon ($r_* \rightarrow -\infty$) like $e^{2\kappa r_*}$, where $\kappa = 2\pi T_H$ is the surface gravity.

Near the horizon (71) becomes

$$\chi'' + \omega_h^2 \chi \simeq 0, \quad (73)$$

up to exponentially decaying terms, where $\omega_h = \omega - k(1 + \frac{r_0^2}{r_n^2})^{-1/2}$. This verifies the claim made earlier that near the horizon the solutions are plane waves in the tortoise coordinate, with frequency ω_h .

The absorption coefficient (34) describes the scattering behavior of the solutions to (71) in the low frequency limit $\omega_\infty r_{max} \ll 1$. At high frequencies when most of the incident wave is absorbed, the Born approximation can be used: $|S|^2 = \frac{1}{4\omega_\infty^2} |\int dy V(y) e^{-2i\omega_\infty y}|^2$, valid when $\omega_\infty^2 > Max(V)$. To find the frequency cutoff, we need the height of the potential barrier. We specialize to the neutral case at this point. In this case V_{coul} vanishes. The result in the charged case is similar.

The gravitational potential V_{grav} is rather complicated and is difficult to maximize analytically. However, it is straightforward to estimate the height. One finds that $Max(V) \sim O(1)r_{max}^{-2}$, and so we need

$$\omega > \omega_{over} \simeq 1/r_{max} \quad (74)$$

Once this condition on the frequency is satisfied, the integral in the Born approximation above goes rapidly to zero with increasing ω : the integrand is an oscillatory factor times the positive definite potential, and one can check that the width of the potential is much greater than the wavelength. Therefore

$$|\mathcal{A}(\omega)|^2 \rightarrow 1, \quad \omega > \omega_{over} \quad (75)$$

This classical result (75) clearly disagrees with the D-brane prediction (17).

It is simple to check that ω_{over} is above the Hawking temperature, their relation being given approximately by

$$\frac{\omega_{over}}{T_H} \simeq \frac{2\pi r_{max} r_n}{r_0^2} \gg 1. \quad (76)$$

Emission is therefore going to zero in this regime. It seems reasonable, however, to expect emission from the D-brane system to approach zero in the same manner. We note that even

though $\omega_{over} \ll T_H$, it may still be small compared to the mass difference $\Delta M = N_R/R$ from extremality. We find that $\omega_{over} \ll \Delta M$ is satisfied if

$$r_{max} \gg \frac{16g^2 r_n^2}{RVr_0^4}. \quad (77)$$

In this case it is appropriate to model emission as a thermal process. The absorption coefficient therefore approaches unity below the energy range in which the considerations of reference [12,16] are necessary.

There appears to be an error then in the D-brane prediction at high frequencies (but frequencies which are still small compared to the string scale, for macroscopic sized black holes). The basic vertex used in the D-brane calculations does not correctly describe the higher energy processes. This vertex is the first term in a low-energy expansion of the exact string vertex in [6]. However, it is not difficult to check that, if one considers the next terms in the expansion, the behavior of the resulting approximate vertex also does not give an absorption coefficient approaching unity. If the D-branes are to give this characteristic black hole behavior in the particle limit, there is some piece of D-brane physics which is missing from the present picture.

7. Conclusions

In this paper, we have shown that the spacetime absorption coefficient has the form found in [7] and given in (34) over a large range of parameter space. Moreover, our derivation gives explicit limits to the range of validity of this result. The spacetime result (34) disagrees in general with the D-brane expression (17). We argued in section 3, based on the conjectured statistical mechanics of D-branes, that agreement should be expected for $r_n \ll r_1, r_5$.

In section 5, we saw that in the neutral case, the results agree over precisely this regime (as we would expect, based on [7]). However, for charged emission, we found that the expressions disagreed for parameters in the FBH regime, in which charge is quantized in much smaller units on the brane than in the bulk of spacetime. Whether, or not, one accepts the argument of section 3, this demonstrates that the current D-brane calculations are at best insufficient to describe charged emission in the FBH regime correctly. It seems plausible that a better understanding of D-brane dynamics will resolve this issue, insofar as in the current model the difference in charge quantization conditions, between the brane

and the bulk of spacetime, prohibits low lying left-moving excitations on the brane from contributing to charge emission.

In section 6, we showed that the Born approximation can be used to show that for $\omega > \omega_{over}$, but still below the string scale, the spacetime absorption coefficient approaches unity, as it should in the particle limit. This behavior, however, is not seen in the D-brane results. The effective vertex used in the D-brane calculation should be valid up to the string scale. This result then poses an additional challenge for the D-brane model.

Note Added: After this work was completed the paper [15] appeared also giving the expression for the spacetime absorption coefficient (34) over the extended parameter region.

In addition, the paper [13] appeared which treats the case of two charges, *e.g.* r_1, r_n , being small. In attempting to reconcile our result with that of [13] we discovered that, in the limit

$$r_1, r_n, r_0 \ll r_5, \tag{78}$$

the requirement for the hypergeometric equation to hold becomes $r \ll r_0$, implying that our result does not hold in this regime. Fortunately, the methods of [13] lead simply to an improved U-duality invariant result. Klebanov and Mathur showed that the C_2 term in (32) may be included in the mapping to the hypergeometric equation. Carrying through the calculation in [13] without making the approximation (78) then gives a result for the absorption coefficient of the form (34) but with $\sqrt{C + D}$ replaced by $\sqrt{C + D + C_2}$ in the coefficients a and b in (34). It is interesting to note that if the result were extended to $\sqrt{C + D + C_2 + C_3}$, then the exponential factor in the numerator of (34) would exactly cancel the Hawking term. Moreover, this modification also correctly yields the exact horizon area in the low energy limit of the absorption coefficient and fits in well with the considerations of [34].

Acknowledgements: We would like to thank J. Gauntlett for useful conversations. DK and JT thank the Aspen Center for Physics, where this project was initiated. FD thanks the Relativity group at DAMTP, Cambridge, for hospitality during the writing up of this work. JT is supported in part by NSF grant NSF-THY-8714-684-A01. FD was supported in part by US DOE grant DE-FG03-92-ER40701 at the Lauritsen Laboratory, Caltech.

Appendix A. Details of Absorption Coefficient Calculations

The solution to the wave equation (32) is constructed by matching three pieces. Near infinity ($v = 0$), the solution is given by a Bessel function (37), near the horizon ($v = 1$) by a hypergeometric function (41), and these are linked together by a “middle solution” (47). The large v (small r) expansion of the Bessel function can be matched to ϕ_{mid} , as is done in (49). The large v expansion of the Bessel function is given in (39), where the coefficients are

$$b_1 = i\sqrt{\frac{2}{\pi}}e^{-i\pi/4}\frac{(1+i\mathcal{S})}{r_0^2\sqrt{\omega_\infty}}, \quad b_3 = i\frac{e^{-i\pi/4}}{2\sqrt{2\pi}}\omega_\infty^{3/2}(1+i\mathcal{S}), \quad (\text{A.1})$$

$$b_2 = \frac{1}{2}e^{-i\pi/4}\sqrt{\frac{\pi}{2}}\omega_\infty^{3/2}\left[(1-i\mathcal{S}) + \frac{i}{\pi}(1+i\mathcal{S})(1-2\gamma+2\ln 2 - \ln \omega_\infty^2 r_0^2)\right]. \quad (\text{A.2})$$

Likewise, the hypergeometric function can be expanded for $v \rightarrow 0$. To derive (43), one can use, as in [7], equation (15.3.6) from Abramowitz and Stegun[35]. (There is a misprint in (4.21) of [7]; the first two arguments of the last hypergeometric function should be interchanged.) This involves inserting a regulator and taking a limit. Alternatively, one can directly use equation (15.3.11) from [35], which gives

$$F(-ia, -ib, 1-ia-ib, 1-v) \rightarrow E\left[1 + v(g - \frac{i}{2}(a+b)) - abv \ln v\right], \quad v \rightarrow 0 \quad (\text{A.3})$$

where g is given in (44), and as found in [7],

$$E(a, b) \equiv \frac{\Gamma(1-ia-ib)}{\Gamma(1-ia)\Gamma(1-ib)}. \quad (\text{A.4})$$

In deriving the form (34) for $|\mathcal{A}|$ one uses (6.1.31) of [35] and finds, as in [7],

$$\frac{1}{|E|^2} = \frac{2\pi ab}{(a+b)} \frac{(e^{2\pi(a+b)} - 1)}{(e^{2\pi a} - 1)(e^{2\pi b} - 1)} \quad (\text{A.5})$$

In working out some of the bounds, and when computing the flux, we use some of the properties of g , defined in (44). Using (6.3.13) of [35] one has

$$\begin{aligned} \text{Im}(g) &= \frac{a+b}{2} - ab \text{Im}[\psi(1-ia) + \psi(1-ib)] \\ &= \pi ab \frac{(e^{2\pi(a+b)} - 1)}{(e^{2\pi a} - 1)(e^{2\pi b} - 1)} \end{aligned} \quad (\text{A.6})$$

And, using (6.3.17) of [35],

$$Re(g(a, b)) = ab \left[1 - a^2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n^2 + a^2} - b^2 \sum_{n=1}^{\infty} \frac{1}{n} \frac{1}{n^2 + b^2} \right] \quad (\text{A.7})$$

So over the range of interest for a, b , we have

$$Re(g(1, 1)) \simeq -\frac{1}{5} \quad (\text{A.8})$$

$$Re(g(a, b)) \simeq ab, \quad a, b \ll 1 \quad (\text{A.9})$$

Lastly, we make some comments about the solution in the middle region and the matching. When the ansatz (47) for ϕ_{mid} is substituted into the wave equation (32), this form of solution requires that terms of order $\frac{C_3 a_0}{v^3}$ have been dropped compared to $\frac{C_2 a_0}{v^2}$. This is consistent for $\frac{r_0^2}{r_1^2} \ll v$. Although this requirement looks like we are losing the overlap region, this does match onto the Bessel function. This is most easily seen by looking at how the analogous expansion works for Bessel's equation: Bessel's equation for the function $f = r^{-3/2} H^{(1,2)}$ written in the variable v is $f''(v) + (\frac{C_3}{v^3} + \frac{C_2}{v^2})f = 0$. Substituting in an expansion of the form (47) gives the same relation for B_0 as in (48).

On the other hand, it can also be seen directly from the differential equations that the coefficient B_1 of the $v \ln v$ term for the actual solution, is different from the coefficient inferred from the hypergeometric equation.

References

- [1] G.T. Horowitz, *The Origin of Black Hole Entropy in String Theory*, to appear in the proceedings of the Pacific Conference on Gravity and Cosmology, gr-qc/9604051 .
- [2] J. Maldacena, *Black Holes in String Theory*, hep-th/9607235 .
- [3] C.G. Callan and J. Maldacena, *D-Brane Approach to Black Hole Quantum Mechanics*, Nucl. Phys. **B472** (1996) 591, hep-th/9602043 .
- [4] A. Dhar, G. Mandal and S.R. Wadia, *Absorption vs. Decay of Black Holes in String Theory and T-Symmetry*, Phys. Lett. **388B** (1996) 51, hep-th/9605234 .
- [5] S.R. Das and S.D. Mathur, *Comparing Decay Rates for Black Holes and D-Branes*, Nucl. Phys. **B478** (1996) 561, hep-th/9606185 .
- [6] A. Hashimoto and I.R. Klebanov, *Decay of Excited D-branes*, Phys. Lett. **381B** (1996) 437, hep-th/9604065 .
- [7] J. Maldacena and A. Strominger, *Black Hole Greybody Factors and D-Brane Spectroscopy*, Phys. Rev. **D55** (1997) 861, hep-th/9609026 .
- [8] S.S. Gubser and I.R. Klebanov, *Four-Dimensional Greybody Factors and the Effective String*, Phys. Rev. Lett. **77**, 4491, hep-th/9609076 .
- [9] C.G. Callan, S.S. Gubser, I.R. Klebanov and A.A. Tseytlin, *Absorption of Fixed Scalars and the D-brane Approach to Black Holes*, hep-th/9610172 .
- [10] S.P. de Alwis and K. Sato, *Radiation from a Class of String Theoretic Black Holes*, hep-th/9611189 .
- [11] I.R. Klebanov and M. Krasnitz, *Fixed Scalar Greybody Factors in Five and Four Dimensions*, hep-th/9612051 .
- [12] E. Keski-Vakkuri and P. Krauss, *Microcanonical D-branes and Back Reaction*, hep-th/9610045 .
- [13] I.R. Klebanov and S.D. Mathur, *Black Hole Greybody Factors and Absorption of Scalars by Effective Strings*, hep-th/9701187.
- [14] J. Maldacena and A. Strominger, *Universal Low-Energy Dynamics for Rotating Black Holes*, hep-th/9702015.
- [15] S.W. Hawking and M.M. Taylor-Robinson, *Evolution of Near Extremal Black Holes*, hep-th/9702045.
- [16] S. Das, A. Dasgupta and T. Sarkar, *High Energy Effects on D-Brane and Black Hole Emission Rates*, hep-th/9702075
- [17] S.R. Das and S.D. Mathur, *Excitations of D-Strings, Entropy and Duality*, Phys. Lett. **375B** (1996) 103, hep-th/9601152 .
- [18] J. Maldacena and L. Susskind, *D-Branes and Fat Black Holes*, Nucl. Phys. **B475** (1996) 670, hep-th/9604042 .
- [19] S.W. Hawking, *Particle Creation by Black Holes*, Comm. Math. Phys. **43** (1975) 199.

- [20] G.T. Horowitz, J. Maldacena and A. Strominger, *Nonextremal Black Hole Microstates and U-Duality*, Phys. Lett. **383B** (1996) 151, hep-th/9603109.
- [21] S.S. Gubser and I.R. Klebanov, *Emission of Charged Particles from Four- and Five-Dimensional Black Holes*, hep-th/9608108 .
- [22] H.F. Dowker, D. Kastor and J.Traschen, unpublished.
- [23] M. Cvetič and D. Youm, *General Rotating, Five Dimensional Black Holes of Toroidally Compactified Heterotic String*, Nucl. Phys. **B476** (1996) 118, hep-th/9603100 .
- [24] G. Horowitz and A. Strominger, *Counting States of Near Extremal Black Holes*, hep-th/9602051 .
- [25] A. Hashimoto, *Perturbative Dynamics of Fractional Strings on Multiply Wound D-Strings*, hep-th/9610250 .
- [26] G. Polhemus, *Statistical Mechanics of Multiply Wound D-Branes*, hep-th/9612130 .
- [27] W.G. Unruh, *Absorption Cross Section of Small Black Holes*, Phys. Rev. **D14** (1976) 325.
- [28] D.N. Page, *Particle Emission Rates from a Black Hole: Massless Particles from an Uncharged Hole*, Phys. Rev. **D13** (1976) 198.
- [29] A.A. Starobinski and S.M. Churilov, *Amplification of Electromagnetic and Gravitational Waves by A Rotating “Black Hole”*, Sov. Phys. JETP **38** (1974) 1.
- [30] S.A. Teukolsky and W.H. Press, *Perturbations of a Rotating Black Hole III: Interaction of the Hole with Gravitational and Electromagnetic Radiation*, Ap. J. **193** (1974) 443.
- [31] G.W. Gibbons, *Vacuum Polarization and the Spontaneous Loss of Charge by Black Holes*, Comm. Math. Phys. **44**, 245 (1975).
- [32] M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, The Parabolic Press, Stanford California (1975).
- [33] S.R. Das, G. Gibbons and S.D. Mathur, *Universality of Low-Energy Absorption Cross-Sections for Black Holes*, Phys. Rev. Lett. **78** (1997) 417, hep-th/9609052 .
- [34] F. Larsen, *A String Model of Black Hole Microstates*, hep-th/9702153 .
- [35] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, National Bureau of Standards New York: Wiley (1964).